Network Formation and Social Norms Under Complentarities: Statics and Dynamics

1 Static Model

1.1 Set Up

- Consider a set of N individuals. Each individual has a type $\theta_i = (\theta_1, ..., \theta_K) \in \Theta$. Individuals know the players.
- Define the *undirected network* $g \in G$ as an $N \times N$ symmetric matrix where $g_{i,j} = g_{j,i} = 1$ there is a link between i and j and $g_{i,j} = g_{j,i} = 0$ otherwise. The space of undirected networks G is the set of all symmetric $N \times N$ matrices with elements in $\{0, 1\}$.
- Define i's *neighbors* $N_i(g) = j \neq i : g_{i,j} = 1$ as the set of individuals with whom i shares a link. i's *degree* $d_i(g) = |N_i(g)|$.
- The game has two periods. In the first, players form links consensually (starting from the empty network). In the second, players choose actions from a set of actions X.
- A strategy of player $i \in N$ is a vector $s_i = (\gamma_i, x_i)$.
  - $\gamma_i = (\gamma_{i,1}, ..., \gamma_{i,i-1}, 1, \gamma_{i,i+1}, ..., \gamma_{i,N})$ with $\gamma_{i,j} \in 0, 1$
  - $x_i = (x_{i,1}, ..., x_{i,K}) \in X \subseteq \Re^K_+$ with $K < \infty$.
- The network $g \in G$ is then created where $g_{ij} = g_{ji} = 1$ if $\gamma_{ij} = \gamma_{ji} = 1$ and $g_{ij} = g_{ji} = 0$ otherwise.
- Utilities take the following form:

$$u_i(x, g) = \underbrace{\theta_i x_i}_{\text{Private benefit}} + \underbrace{\left[a x_i \sum_{j \in N_i(g)} x_j \right]}_{\text{Complementarities}} - \underbrace{c_x(x_i)}_{\text{Cost of X}} - \underbrace{c_d d_i(g)}_{\text{Cost of Links}}$$

  where

  - a is a K-vector of complentarities
  - $c_d(d_i(g))$ is a cost function $c_d : G \to \Re_+$. $d_i(g)$ is the *degree* (number of links i has) given a network g.
  - $c_X(x_i)$ is a cost function $c_X : X \to \Re_+$

1.2 Equilibrium Concept(s)

- We are interested in two equilibrium concepts.

  1. **Pairwise Stability:** No two nodes i and j with $g_{i,j} = 0$ are better off forming a link and no two nodes i and j with $g_{i,j} = 1$ are better of breaking their link.
  2. **Efficiency:** $U(x, g) = \Sigma_i u_i(x, g) \geq \Sigma_i u_i(x', g') = U(x', g')\ \forall x' \in X, \forall g' \in G$

1.3 Motivating "Simple" Example

There are ten individuals ($N = 10$) and two types ($|\Theta| = 2$). Individuals of the first type have $\theta_{i,1} > \theta_{i,2}$ and individuals of the second type have $\theta_{i,1} < \theta_{i,2}$. Let the action space $X = \{0, 1\}^2$ consist of two possible actions: $x_{i,1}$ is going to the bar on Friday and $x_{i,2}$ is going to the bar on Saturday...

2 Dynamic Model

2.1 Set Up

- Consider a set of N *infinitely-lived individuals.* Each individual has a *type* $\theta_i = (\theta_1, ..., \theta_K) \in \Theta$. At $t = 0$, individuals do not know the players.

- Define the *undirected network* $g(t) \in G$ as an $N \times N$ symmetric matrix where $g_{i,j}(t) = g_{j,i}(t) = 1$ there is a link between i and j and $g_{i,j}(t) = g_{j,i}(t) = 0$ otherwise. The space of undirected networks G is the set of all symmetric $N \times N$ matrices with elements in $\{0, 1\}$.

- Define i's *neighbors* $N_i(g(t)) = j \neq i : g_{i,j}(t) = 1$ as the set of individuals with whom i shares a link. i's *degree* $d_i(g(t)) = |N_i(g(t))|$. Individuals observe the actions $x_j(t)$ of their neighbors.

  - When individuals i and j meet in period t, they learn the action choices $x_i(t-1)$ and $x_j(t-1)$.

- Let $p(g_{ij}(t-1), x_i(t-1), x_j(t-1))$ represent the *probability that individuals i and j meet in period t*, which is a function of their action choices $x_i(t-1)$ and $x_j(t-1)$ from the previous period. If $g_{ij}(t-1) = 1$, $p(\cdot) = 0$

- In each period t, individuals first "meet" a random subset $n_i(t) \subseteq N$ where $i \in n_j(t) \iff j \in n_i(t)$. They can choose to make links with the people they meet, and break any existing links. Then, players choose from a set of actions X.

- A *strategy* of player $i \in N$ in period t is a vector $s_i(t) = (\gamma_i(t), x_i(t))$.

- $\gamma_i(t) = (\gamma_{i,1}(t), ..., \gamma_{i,i-1}(t), 1, \gamma_{i,i+1}(t), ..., \gamma_{i,N}(t))$ with $\gamma_{i,j}(t) \in \{0,1\}$ and $\gamma_{i,j}(t) - \gamma_{i,j}(t-1) > 0 \implies j \in n_i(t)$
  - Individuals can only form links with people they “meet” in a period t, but can break links at any time.
- $x_i(t) = (x_{i,1}(t), ..., x_{i,K}(t)) \in X \subseteq \Re^K_+$ with $K < \infty$.

- The network $g(t) \in G$ is then created where $g_{ij}(t) = g_{ji}(t) = 1$ if $\gamma_{ij}(t) = \gamma_{ji}(t) = 1$ and $g_{ij}(t) = g_{ji}(t) = 0$ otherwise.
- Utilities in each period take the same form, but are indexed by time:

$$U_i(x(t), g(t), t) = \underbrace{\theta_i x_i(t)}_{\text{Private benefit}} + \underbrace{\left[a x_i(t) \sum_{j \in N_i(g(t))} x_j(t) \right]}_{\text{Complementarities}} - \underbrace{c_x(x_i(t))}_{\text{Cost of X}} - \underbrace{c_d d_i(g(t))}_{\text{Cost of Links}}$$

where

- a is a K-vector of complentarities
- $c_d(d_i(g(t)))$ is a cost function $c_d : G \to \Re_+$. $d_i(g)$ is the *degree* (number of links i has) given a network g.
- $c_X(x_i(t))$ is a function $c_X : X \to \Re^+$

$\vdots$

Period t: Meeting set $n_i(t)$ is revealed

$\searrow$

Individuals choose $\gamma_i(t)$

$\searrow$

Network $g(t)$ is formed

$\searrow$

Individuals choose action $x_i(t)$

$\searrow$

Utility $U_i(t)(x(t), g(t))$ realized

Period t+1:

$\vdots$

Figure 1: Game Tree for One Period

2.2 Equilibrium Concept(s)

- We are interested in two equilibrium concepts.

  1. **Pairwise Stability:** No two individuals i and j with $g_{i,j}(t) = 0$ are better off forming a link and no two nodes i and j with $g_{i,j}(t) = 1$ are better of breaking their link.
     - **(Feasible) Pairwise Stability** is a weakening of this equilibrium concept where we require pairwise stability only for individuals who have a non-zero probability of meeting.
  2. **Efficiency:** $U(x(t), g(t), t) = \Sigma_i u_i(x(t), g(t), t) \geq \Sigma_i u_i(x', g', t) = U(x', g', t)$ $\forall x' \in X$, $\forall g' \in G$